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Operator Algebras and the Conjugacy of Transformations

DONALD W. HADWIN

Department of Mathematics, University of New Hampshire, Durham, New Hampshire 03824

AND

THOMAS B. HOOVER

Department of Mathematics, University of Hawaii, Honolulu, Hawaii 96822

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A family of algebras is associated with each continuous map on a compact Hausdorff space. These algebras are called conjugacy algebras. It is shown that two continuous maps are conjugate if and only if some conjugacy algebra of one is isomorphic to some conjugacy for the other. Numerous examples of conjugacy algebras are provided. © 1988 Academic Press, Inc.

1. INTRODUCTION

In [1] W. B. Arveson associates a Banach algebra $A(X, m, \eta)$ of operators on $L^2(m)$ with each triple (X, m, η), where (X, m) is a probability space and η is an ergodic, measure-preserving $*$-automorphism of $L^\infty(m)$. Arveson proved [1, Theorem 1.8] that two triples (X_1, m_1, η_1) and (X_2, m_2, η_2) are conjugate (i.e., there is a measure-preserving $*$-automorphism τ: $L^\infty(m_2) \to L^\infty(m_1)$ with $\tau \circ \eta_2 = \eta_1 \circ \tau$) if and only if the algebras $A(X_1, m_1, \eta_1)$ and $A(X_2, m_2, \eta_2)$ are unitarily equivalent. Later, Arveson and K. B. Josephson [2] extended Arveson's result by considering pairs (X, η), where X is a locally compact Hausdorff space and η is a homeomorphism on X, with the additional property that there is a regular Borel probability measure m on X such that

(1) $m \circ \eta$ and m are mutually absolutely continuous,

(2) $m(V) > 0$ for every non-empty open set V, and

(3) $m(\bigcup_{n \geq 0} \{x \in X: \eta^n(x) = x\}) = 0$.

Such measures need not exist [3, 6]. Arveson and Josephson associate a Banach algebra $A(X, \eta)$ of operators on $L^2(m)$ with each such pair (X, η);

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they show [2] that the algebra is, up to isomorphism, independent of the measure m. Then they prove [2, Theorem 3.11] that two pairs (X_1, η_1) and (X_2, η_2) are conjugate (i.e., there is a homeomorphism $\tau: X_2 \to X_1$ such that $\tau \circ \eta_2 = \eta_1 \circ \tau$) if and only if the algebras $A(X_1, \eta_1)$ and $A(X_2, \eta_2)$ are isomorphic, assuming that each of the corresponding measures can be chosen to be both ergodic and invariant. This generalizes [1] via Gelfand theory.

The algebra $A(X, \eta)$ is canonically isomorphic to a subalgebra of the crossed product C^*-algebra $C^*(X, \eta)$ (cf. [2, 4] for definitions and references); this algebra is called the semi-crossed product of $C(X)$ with η in [5]. The semi-crossed product C^*-algebra is defined when η is a continuous map on a compact Hausdorff space X, and does not depend on the existence of a measure satisfying (1)–(3) above. J. Peters [5, Theorem V.1] showed that if one uses the semi-crossed product, the Arveson–Josephson theorem extends to the case when the X's are compact and the η's are continuous and aperiodic.

In this paper, a family of algebras is associated with each pair (X, η), where $\eta: X \to X$ is a continuous map on the compact Hausdorff space X. These algebras are called conjugacy algebras for the pair (X, η). It is then shown (Theorem 3.1) that two pairs (X_1, η_1) and (X_2, η_2) are conjugate (in the Arveson–Josephson sense) if and only if some conjugacy algebra for (X_1, η_1) is isomorphic to some conjugacy algebra for (X_2, η_2), with certain restrictions on the sets of fixed points of η_1 and η_2. The Arveson–Josephson algebras [2] and the semi-crossed product algebras [5] are conjugacy algebras, so this paper generalizes both [2] and [5] in various ways. First, the only periodicity assumptions have to do with the fixed points of the continuous maps. More significantly, since each pair (X, η) has many non-isomorphic conjugacy algebras, we obtain a variety of theorems of Arveson–Josephson type. In addition, while Arveson and Josephson [2] use deep properties of von Neumann algebras, our results rely only upon elementary (mostly algebraic) techniques. In fact, when the continuous maps are free of fixed points, our proof becomes very simple (see Section 3).

2. Conjugacy Algebras

Before defining conjugacy algebras, we will consider two extremal examples. Let $C(X)$ denote the continuous complex functions on X, and let $P(X, \eta)$ denote the polynomials in the variable U with coefficients in $C(X)$. We define addition in $P(X, \eta)$ as the usual addition of polynomials, but multiplication is defined so that $Uf = (f \circ \eta)\, U$ for every f in $C(X)$. We call $P(X, \eta)$ the *skew polynomial algebra* on (X, η). Similarly, we define the algebra $P^{\infty}(X, \eta)$ of *skew formal power series* on (X, η) with the same "twisted" multiplication. A conjugacy algebra for (X, η) will contain a copy of $P(X, \eta)$ in such a way that the inclusion map from $P(X, \eta)$ into $P^{\infty}(X, \eta)$ can be extended to a homomorphism from the conjugacy algebra into $P^{\infty}(X, \eta)$. It is through this extended homomorphism that each element a in the conjugacy algebra is associated with a formal power series $\sum_n \Pi_n(a)\, U^n$ in $P^{\infty}(X, \eta)$. A crucial property of the skew multiplication in $P^{\infty}(X, \eta)$ is its relation to the fixed points of the function η. More precisely, if $x \in X$, then the map $\sum_n f_n U^n \to \sum_n f_n(x)\, z^n$ from $P^{\infty}(X, \eta)$ into the ring $\mathbb{C}[\![z]\!]$ of complex formal power series is a homomorphism if and only if $\eta(x) = x$. It is the notion that these complex formal power series are like analytic functions, in the sense that the "real-valued" ones are constant (see Lemma 2.3), that motivates part (4) of the definition below.

DEFINITION 2.1. Let X be a compact Hausdorff space and η: $X \to X$ be continuous. A complex algebra A is a *conjugacy algebra* for (X, η)

(1) if $P(X, \eta)$ is contained in A and the constant 1 function is the identity for A;

(2) if there is an algebra homomorphism Π_0: $A \to C(X)$ such that

(a) $\Pi_0(f) = f$ for every f in $C(X)$,

(b) $\ker(\Pi_0) = AU$;

(3) if U is not a right divisor of zero; and

(4) if Y is a compact Hausdorff space, β: $C(Y) \to A$ is an algebra homomorphism, and α is a multiplicative linear functional on A, then $\alpha \circ \beta = \alpha \circ \Pi_0 \circ \beta$.

In actuality, we will not insist that a conjugacy algebra contain $P(X, \eta)$, but rather an isomorphic copy of $P(X, \eta)$ in such a way that the spirit of (1)–(4) is maintained. We now show how to define a homomorphism from a conjugacy algebra A into $P^{\infty}(X, \eta)$. We wish to associate a formal power series $\sum_n \Pi_n(a)\, U^n$ with each a in A. The map Π_0 is given in (2b) above. Since $a - \Pi_0(a) \in \ker(\Pi_0) = AU$, we can write $a = \Pi_0(a) + bU$ for some b in A. Since U is not a right divisor of zero, b is uniquely defined. We can now inductively define the coefficient maps $\Pi_1, \Pi_2, \ldots$ by $\Pi_{n+1}(a) = \Pi_n(b)$. Direct computation shows that the map $a \to \sum_n \Pi_n(a)\, U^n$ is indeed a homomorphism from A into $P^{\infty}(X, \eta)$ whose restriction to $P(X, \eta)$ is the inclusion map.

We spend the remainder of this section providing examples of conjugacy algebras for a given pair (X, η). First we shall make precise our remark that "real-valued" complex formal power series are constant.

LEMMA 2.3. *Suppose Y is a compact Hausdorff space and $h: C(Y) \to \mathbb{C}[[z]]$ is a homomorphism. Then the range of h is contained in the scalars.*

Proof. Write $h(f) = \sum_n h_n(f) z^n$ for each f in $C(Y)$. We wish to show that the linear functionals h_n vanish when $n > 0$. Since h is a homomorphism, we have $h_1(fg) = h_0(f) h_1(g) + h_1(f) h_0(g)$ for all f, g in $C(Y)$. It follows that h_1 vanishes on the squares of elements in $\ker(h_0)$. Since h_0 is a complex homomorphism on $C(Y)$, it follows that every element in $\ker(h_0)$ is a linear combination of squares in $\ker(h_0)$; whence, h_1 vanishes on $\ker(h_0)$. Since $h_1(1) = 0$ and $C(Y)$ is spanned by 1 and $\ker(h_0)$, we conclude that $h_1 = 0$. Now it follows that $h_2(fg) = h_0(f) h_2(g) + h_2(f) h_0(g)$ for all f, g in $C(Y)$. Proceeding inductively, we conclude that $h_n = 0$ for $n > 0$.

If A is a conjugacy algebra for (X, η) and $x \in X$ is a fixed point for η, then the map $a \to \sum_n \Pi_0(a)(x) z^n$ is a homomorphism from A into $\mathbb{C}[[z]]$ whose kernel is an ideal we shall denote by J_x. More precisely, $J_x = \{a \in A: \Pi_n(a)(x) = 0 \text{ for } n = 0, 1, \ldots\}$. Note that if x is not a fixed point, then the latter set is not an ideal of A.

If A is a conjugacy algebra for (X, η), let $M = M(A)$ denote the set of non-zero multiplicative linear functionals on A. For each α in M, there is an x in X such that $\alpha(f) = f(x)$ for every f in $C(X)$. Let M_x denote the set of all α in M such that $\alpha(f) = f(x)$ for every f in $C(X)$. No M_x is empty since $\alpha_x(a) = \Pi_0(a)(x)$ defines a complex homomorphism. The following result shows that part (4) of Definition 2.1 is related only to the fixed points of η.

PROPOSITION 2.4. *Suppose $\eta: X \to X$ is a continuous function on the compact Hausdorff space X, and let A be an algebra satisfying conditions* (1)–(3) *of Definition* 2.1. *Then*

(1) *if x is not a fixed point η, then $M_x = \{\alpha_x\}$;*

(2) *if η has no fixed points, then A is a conjugacy algebra for $\{X, \eta)$;*

(3) *if $J_x \subseteq \ker(\alpha)$ for every α in M_x whenever x is a fixed point for η, then A is a conjugacy algebra for (X, η).*

Proof. (1) Suppose $\eta(x) \neq x$, and choose an f in $C(X)$ such that $f(x) \neq f(\eta(x))$. If $\alpha \in M_x$, then $0 = \alpha(Uf - fU) = \alpha(U)(f(x) - f(\eta(x)))$; whence, $\alpha(U) = 0$. Thus $\ker(\Pi_0) \subseteq \ker(\alpha)$, which implies $\alpha(a) = (\Pi_0(a)) = \Pi_0(a)(x) = \alpha_x(a)$ for every a in A.

(2) Since $\alpha = \alpha \circ \Pi_0$ for every α in M (by (1) above), it follows that (4) in Definition 2.1 holds, and therefore A is a conjugacy algebra for (X, η).

(3) Suppose x is a fixed point for η, and $\alpha \in M_x$. Let $h: A \to \mathbb{C}[[z]]$ be the homomorphism defined by $h(a) = \sum_n \Pi_n(a)(x) z^n$. Then $\ker(h) = J_x \subseteq \ker \alpha$. Thus there is a complex homomorphism g defined on the range of h so that $\alpha = g \circ h$. Suppose Y is a compact Hausdorff space and $\beta: C(Y) \to A$ is a homomorphism. It follows from Lemma 2.3 that $(h \circ \beta)(f) = \Pi_0(\beta(f))(x) = h(\Pi_0(\beta(f)))$ for every f in $C(Y)$. Hence $\alpha \circ \beta = \alpha \circ \Pi_0 \circ \beta$. Thus part (4) of Definition 2.1 holds, which implies that A is a conjugacy algebra for (X, η).

COROLLARY 2.5. *$P(X, \eta)$ is a conjugacy algebra for (X, η).*

LEMMA 2.6. *$P^\infty(X, \eta)$ is a conjugacy algebra for (X, η).*

Proof. A simple computation shows that an element a of $P^\infty(X, \eta)$ is invertible if and only if $\Pi_0(a)$ is invertible. It follows that if $\alpha \in M_x$ for some x in X, then $\Pi_0(\ker \alpha)$ is a proper ideal in $C(X)$ containing $\{f \in C(X): f(x) = 0\}$; whence, applying Π_0^{-1}, we see that $\ker \alpha \subseteq \ker(\alpha_x)$. It follows that $\alpha = \alpha_x$. Thus part (4) of Definition 2.1 holds (parts (1)–(3) are obvious), which implies that $P^\infty(X, \eta)$ is a conjugacy algebra for (X, η).

The remaining examples of conjugacy algebras will all be Banach algebras. If A is a Banach algebra satisfying conditions (1)–(3) of Definition 2.1, then the map $\Pi_0: A \to A$ need not be continuous on A; it follows from the open mapping theorem that the map is continuous if $C(X)$ is a closed subspace of A. In this case all the coefficient maps are continuous. To see this, note that the map $a \to aU$ is continuous and injective on A and has closed range (equal to $\ker \Pi_0$). It follows from the open mapping theorem that this map has a continuous inverse T; $T(aU) = a$. Then $\Pi_n = \Pi_0(T(1 - \Pi_0))^n$ is bounded for each n.

PROPOSITION 2.7. *Suppose that A is a Banach algebra satisfying conditions* (1)–(3) *of Definition* 2.1 *such that $C(X)$ is closed in A and $P(X, \eta)$ is norm dense in A, and such that* $\limsup_n [\|\Pi_n\| \, \|U^n\|]^{1/n} \leqslant 1$. *Then A is a conjugacy algebra. Furthermore, if x is a fixed point of η, then M_x can be naturally identified with the spectrum of U, which is a closed disk centered at* 0 *with positive radius.*

Proof. Let $r = \lim_n \|U^n\|^{1/n}$ be the spectral radius of U. Since the map $a \to aU$ is injective on A and has closed range, it follows that there is a positive number s such that $\|aU\| \geqslant s\|a\|$ for every a in A. It follows that $\|U^n\| \geqslant s^n \|1\|$ for $n \geqslant 1$. Thus $r \geqslant s > 0$.

It follows from the hypothesis that $\limsup_n \|\Pi_n\|^{1/n} \leqslant 1/r$. Suppose that x is a fixed point and $a \in A$. Then the radius of convergence of the power series $\sum_n \Pi_n(a)(x) z^n$ is at least r, since $\limsup_n |\Pi_n(a)(x)|^{1/n} \leqslant$

$\limsup_n \|\Pi_n\|^{1/n} \leqslant 1/r$. Hence, for every complex number z with $|z| < r$, the mapping $a \to \sum_n \Pi_n(a)(x) z^n$ is a multiplicative linear functional on A, which must therefore have norm 1. Thus, for each a in A and z with $|z| < r$, we have $|\sum_n \Pi_n(a)(x) z^n| \leqslant \|a\|$. Since each a in A is a limit of elements of $P(X, \eta)$, we see that the analytic function $\sum_n \Pi_n(a)(x) z^n$ is a uniform limit of polynomials, and can thus be extended to a continuous function $G(a)$ on the closed disk D centered at 0 with radius r. Hence, for each z in D, the map $a \to G(a)(z)$ is a multiplicative linear functional in M_x that sends U to z. Since $P(X, \eta)$ is dense in A and every functional in M_x is continuous, it follows that each functional in M_x is completely determined by what it does to U. Since the image under an element of M_x must be contained in the spectrum of U, which is contained in D, it follows that the elements in M_x are precisely the maps $a \to G(a)(z)$ for each z in D. Thus the spectrum of U is D, and since $\ker G \subseteq J_x$, it follows from part (3) of Proposition 2.4 that A is a conjugacy algebra.

Remark. The algebras in [1, 2, 5] satisfy the hypothesis of Proposition 2.7, since, in these algebras, U, Π_0, Π_1, ... all have norm 1. Moreover, the analogous algebras defined on L^p-spaces can be shown to be conjugacy algebras in a similar manner.

EXAMPLE. Let W be a Banach space and let $f \to M_f$ be a faithful continuous representation of $C(X)$ as operators on W. For example, W could be $C(X)$ and M_f could be "multiplication by f." Or, if m is a Borel measure on X with $m(V) > 0$ for every non-empty open set V, and if $1 \leqslant q \leqslant \infty$, then we could let W be $L^q(X, m)$ and let M_f be multiplication by f. Fix p, $1 \leqslant p < \infty$, and let Y be the Banach space of all norm p-summable sequences of points in W. Let U be the backwards shift operator on Y, i.e., $U(\{w_n\}) = \{v_n\}$, where $v_n = w_{n+1}$ for $n = 0, 1, \ldots$. For f in $C(X)$, let T_f be the operator on Y defined by $T_f(\{w_n\}) = \{M_{f \circ \eta^n}(w_n)\}$. It is readily verified that the norm closed algebra A generated by U and all the T_f's satisfies the hypothesis of Proposition 2.7, and is therefore a conjugacy algebra for (X, η). The semi-crossed products in [5] are isomorphic to those with W a Hilbert space and $p = 2$.

EXAMPLE. This example is closely related to the algebras in [1, 2]. The verification that they are, indeed, conjugacy algebras is quite like the argument in [1].

Suppose that the continuous map η is freely acting on X in the sense that, for every non-empty open set V and for every positive integer n, there is a non-empty open subset V' of V such that the sets $\eta^k(V')$, $0 \leqslant k \leqslant n$, are pairwise disjoint. It is easily shown that η acts freely if and only if, for each positive integer n, the set $\{x : \eta^n(x) = x\}$ has empty interior.

Suppose m is a Borel measure on X such that $m(V)>0$ for every non-empty open set V, and such that m and $m\circ\eta$ are mutually absolutely continuous with Radon–Nikodym derivative $h=dm\circ\eta/dm$. Also assume that every set with positive measure contains a set with finite positive measure.

Suppose $1\leqslant p\leqslant\infty$, and let $C(X)$ act on $L^p(m)$ as multiplications. Define the operator U on $L^p(m)$ by $U(f)=(f\circ\eta)\,h^{1/p}$. In the case $p=\infty$, let the "weight" function $h^{1/p}$ be the constant 1 function. Let A be the norm closure of $P(X,\eta)$ in the algebra $B(L^p(m))$ of all operators on $L^p(m)$. Note that if m is counting measure and $p=\infty$, then $C(X)$ (with the supremum norm) is a norm closed subspace of $L^\infty(m)$ that is invariant for the algebra A. In this case, once we prove that A is a conjugacy algebra, it will follow that the restriction of A to $C(X)$ is also a conjugacy algebra.

The operator U is an invertible isometry on $L^p(m)$, so $\|U\|=1$. Suppose that $a\in P(X,\eta)$ and $a=\sum_{i=0}^n f_i U^i$. Suppose $0\leqslant k\leqslant n$ and $r>0$, and let $V=\{x\in X\colon |f_k(x)|>\|f_k\|-r\}$. Choose a non-empty open subset V' of $\eta^k(V)$ such that the sets $\eta^i(V')$, $0\leqslant i\leqslant n$, are pairwise disjoint. Let f be the characteristic function of a subset E of V' with $0<m(E)<\infty$. The functions $f\circ\eta^i$, $0\leqslant i\leqslant n$, have pairwise disjoint supports and $U^k f$ vanishes off V. Thus $\|af\|\geqslant\|(f\circ\eta^k)\,af\|=\|f_k(f\circ\eta^k)\,U^k f\|\geqslant(\|f_k\|-r)\,\|f\|$. Since r can be chosen to be arbitrarily small, it follows that $\|a\|\geqslant\|f_k\|$ for $0\leqslant k\leqslant n$. Hence the coefficient maps $\Pi_0, \Pi_1, \ldots$ are continuous with norm 1 on $P(X,\eta)$. Thus the coefficient maps can be extended to contractive linear maps on A. It follows from Proposition 2.7 that A is a conjugacy algebra for (X,η).

3. The Main Results

We are now ready to prove the main result of this paper.

THEOREM 3.1. *Suppose that $\eta_i\colon X_i\to X_i$ is a continuous map on the compact Hausdorff space X_i for $i=1,2$. Suppose also that $\{x\in X_2\colon \eta_2(x)\neq x, \eta_2^2(x)=\eta_2(x)\}$ has empty interior. Then (X_1,η_1) and (X_2,η_2) are conjugate if and only if some conjugacy algebra for (X_1,η_1) is isomorphic to some conjugacy algebra for (X_2,η_2).*

Remarks. (1) Note that the above theorem does not require that every conjugacy algebra for (X_1,η_1) be isomorphic to every conjugacy algebra for (X_2,η_2); indeed, the examples of the preceding section suggest that a pair (X,η) can have many non-isomorphic conjugacy algebras.

(2) We believe that the condition on the fixed points of the η_2 in the above theorem is unnecessary. Note that this condition is satisfied when η_2 is injective or when η_2 has no fixed points. A simple example of a function not satisfying this condition is $\eta(x)=|x|$ on $[-1,1]$.

It is easily seen that if (X_1, η_1) and (X_2, η_2) are conjugate, then $P(X_1, \eta_1)$ and $P(X_2, \eta_2)$ are isomorphic conjugacy algebras. Thus it remains to show that if A_i is a conjugacy algebra for (X_1, η_i) for $i=1, 2$, and if A_1 is isomorphic to A_2, then (X_1, η_1) and (X_2, η_2) are conjugate. We will let U_1 and U_2 denote the distinguished elements ("U" of Definition 2.1) of A_1 and A_2, respectively. We will let $\Pi_0, \Pi_1, \ldots$ denote the coefficient maps in both A_1 and A_2. Let $T: A_1 \to A_2$ be an isomorphism.

Define algebra homomorphisms $L_1: C(X_1) \to C(X_2)$ and $L_2: C(X_2) \to C(X_1)$ by $L_1(f) = \Pi_0(T(f))$ and $L_2(g) = \Pi_0(T^{-1}(g))$. Thus there are continuous maps $\tau_1: X_2 \to X_1$ and $\tau_2: X_1 \to X_2$ such that $L_1(f)(y) = f(\tau_1(y))$ for all f in $C(X_1)$ and y in X_2, and such that $L_2(g)(x) = g(\tau_2(x))$ for all g in $C(X_2)$ and x in X_1. The symmetry in the definitions of L_1, L_2, and τ_1, τ_2 will be used in the proof.

LEMMA 3.2. *$L_2 = L_1^{-1}$ and $\tau_2 = \tau_1^{-1}$.*

Proof. Suppose $y \in X_2$ and α is a multiplicative linear functional in M_y, and let $\beta = \alpha \circ T$. It follows from part (4) of Definition 2.1 that $\beta = \alpha \circ L_1$ on $C(X_1)$ and $\beta \circ T^{-1} = \beta \circ L_2$ on $C(X_2)$. Thus $L_1(L_2(g))(y) = \alpha(L_1(L_2(g)) = \alpha(\Pi(T(L_2(g)))) = \alpha(T(L_2(g))) = \beta(L_2(g)) = \beta(T^{-1}(g)) = \alpha(g) = g(y)$, for every g in $C(X_2)$. A similar argument shows that $L_2(L_1(f)) = f$ for every f in $C(X_1)$. The assertion about the τ_i's is now clear.

Write $\tau = \tau_1$. We shall show that τ is the homeomorphism that implements the conjugacy between (X_1, η_1) and (X_2, η_2). All that remains is showing that $\eta_1 \circ \tau = \tau \circ \eta_2$, or, equivalently, that $f \circ \tau \circ \eta_2 - f \circ \eta_1 \circ \tau = 0$ for every f in $C(X_1)$. To this end, suppose that $f \in C(X_1)$ and let $h = f \circ \tau \circ \eta_2 - f \circ \eta_1 \circ \tau$. We first show that h vanishes on the fixed points of η_2. For $i = 1, 2$, let F_i be the set of fixed points on η_i. Certainly, $\eta_i(F_i) = F_i$.

LEMMA 3.3. *$\tau(F_2) = F_1$ and $h = 0$ on F_2.*

Proof. It is clear from the definition of h that if $\tau(F_2) = F_1$, then $h = 0$ on F_2. Suppose $y \in F_2$ and define the homomorphism $\sigma: A_1 \to \mathbb{C}[[z]]$ by $\sigma(a) = \sum_n \Pi_n(T(a))(y)\, z^n$. It follows from Lemma 2.3 that $\sigma(f) = \Pi_0(T(f))(y) = f(\tau(y))$ for every f in $C(X_1)$. Furthermore, if $\sigma(U_1) = 0$ then $A_1/\ker \sigma$ is 1-dimensional. However, $\ker \sigma = T^{-1}(J_y)$, where $J_y = \{a \in A_2 : \Pi_n(a)(y) = 0,\ n = 0, 1, \ldots\}$ and A_2/J_y is not 1-dimensional. Hence $\sigma(U_1) \neq 0$. Since $\mathbb{C}[[z]]$ is commutative, it follows, for every f in $\mathbb{C}(X_1)$, that $0 = \sigma(fU_1 - U_1 f) = \sigma(f - f \circ \eta_1)\ \sigma(U_1) = (f(\tau(y)) - f(\eta_1(\tau(Y))))\, \sigma(U_1)$. Since $\sigma(U_1) \neq 0$, it follows that $\tau(y) \in F_1$. Hence $\tau(F_2) \subseteq F_1$. The reverse inclusion follows from the symmetry between τ_2 and $\tau_1 = \tau$.

LEMMA 3.4. *If* $y \in X_2 \backslash F_2$, *then* $\Pi_0(T(U_1))(y) = \Pi_0(T(U_1))(\eta_2(y)) = 0$. *Similarly*, $\Pi_0(T^{-1}(U_2)) = 0$ *off* F_1.

Proof. Let α denote the multiplicative linear functional on A_1 given by $\alpha(a) = \Pi_0(T(a))(y)$. Then $\alpha(f) = \Pi_0(T(f))(y) = f(\tau(y))$ for each f in $C(X_1)$. Hence $\alpha \in M_{\tau(y)}$. It follows from Proposition 2.4(1) that $\alpha(a) = \Pi_0(a)(\tau(y))$ for all a in A_1 (since $\tau(y) \notin F_1$). In particular, $\Pi_0(T(U_1))(y) = \alpha(U_1) = 0$. Thus $\Pi_0(T(U_1))$ vanishes on $X_2 \backslash F_2$. The hypothesis on the fixed points in Theorem 3.1 says that $\{y \in X_2 \backslash F_2 : \eta_2(y) \in X_2 \backslash F_2\}$ is dense in $X_2 \backslash F_2$. It follows from continuity that $\Pi_0(U_1) \circ \eta_2$ vanishes on $X_2 \backslash F_2$.

LEMMA 3.5. $\Pi_1(T(U_1))\, h = \Pi_1(T(f \circ \eta_1))\, \Pi_0(T(U_1) \circ \eta_2 - \Pi_0(T(U_1))$ $\Pi_1(T)(f))$.

Proof. From the definition of Π_1 and Π_0, it follows that $\Pi_1(T(U_1 f)) = \Pi_0(T(U_1))\, \Pi_1(T(f)) + \Pi_1(T(U_1))(f \circ \tau \circ \eta_2)$ and $\Pi_1(T(U_1 f)) = \Pi_1(T((f \circ \eta_1)\, U_1)) = (f \circ \eta_1 \tau)(\Pi_1(T(U_1))) + \Pi_1(T(f \circ \eta_1))\, \Pi_0(T(U_1) \circ \eta_2)$. But $U_1 f = (f \circ \eta_1)\, U_1$, so equating the right-hand sides of the preceding equations gives the desired result.

The following lemma completes the proof of Theorem 3.1.

LEMMA 3.6. $h = 0$.

Proof. Since $h \circ \tau^{-1}$ vanishes on F_1 and $\Pi_0(T^{-1}(U_2))$ vanishes off F_1, we have $\Pi_0(L_2(h)\, T^{-1}(U_2)) = (h \circ \tau^{-1})(\Pi_0(T^{-1}(U_2)) = 0$. Hence $L_2(h)\, T^{-1}(U_2) = L_2(h)[T^{-1}(U_2) - \Pi_0(T^{-1}(U_2))] = L_2(h)\, aU_1$ for some a in A_1. Thus $h = \Pi_0((T(L_2(h))) = \Pi_1(T(L_2(h))\, U_2) = \Pi_1(T(L_2(h)\, aU_1)) = h(\Pi_1(T(aU_1)) + \Pi_1(T(L_2(h)))(\Pi_0(T(a)) \circ \eta_2)(\Pi_0(T(U_1)) \circ \eta_2) = h\Pi_1(T(a))(\Pi_0(T(U_1)) \circ \eta_2) + h\Pi_0(T(a))\, \Pi_1(T(U_1)) + \Pi_1(T(L_2(h))$ $(\Pi_0(T(a) \circ \eta_2)(\Pi_0(T(U_1)) \circ \eta_2) = h\Pi_1(T(a))(\Pi_0(T(U_1)) \circ \eta_2) + \Pi_0(T(a))$ $[\Pi_1(T(f \circ \eta_1)(\Pi_0 T(U_1)) \circ \eta_2) - \Pi_0(T(W_1))\, \Pi_1(T(f))] + \Pi_1(T(L_2(h)))$ $(\Pi_0(T(a)) \circ \eta_2)(\Pi_0(T(U_1)) \circ \eta_2)$. The last equality follows from Lemma 3.5. Note that each term of the last expression above has as a factor either $\Pi_0(T(U_1))$ or $\Pi_0(T(U_1)) \circ \eta_2$. But Lemma 3.4 says that each of these vanish off F_2. Hence $h = 0$ off F_2. But Lemma 3.3 says that $h = 0$ on F_2. Thus $h = 0$, and the proof is complete.

We now outline how simple the proof of Theorem 3.1 becomes if we assume that the maps η_1 and η_2 have no fixed points. In this case it follows from Proposition 2.4(1) that $A_i U_i$ is the intersection of the kernels of all the complex homomorphisms on A_i, $i = 1, 2$. It follows that $T(A_1 U_1) = A_2 U_2$, and thus T induces an isomorphism T' from $A_1/A_1 U_1$ to $A_2/A_2 U_2$. However, $A_i/A_i U_i = A_1/\ker \Pi_0$ is isomorphic to $C(X_i)$. Thus there is a homeomorphism $\tau: X_2 \to X_1$ such that $(T(f)-f\circ\tau)\in A_2U_2$, i.e., $\Pi_0(T(f))=f\circ\tau$, for every f in $C(X_1)$. Note that $T(U_1)=VU_2$ and $T^{-1}(U_2)=WU_1$ for some V in A_2 and some W in A_1. Hence $U_2=T(WU_1)=T(W)\,T(U_1)=T(W)\,VU_2$. Since U_2 is not a right divisor of zero, we conclude that V is left invertible in A_2; hence $\Pi_0(V)$ is invertible in $C(X_2)$. Suppose $f\in C(X_1)$. Since $U_1 f=(f\circ\eta_1)\,U_1$, it follows that $VU_2T(f)=T(f\circ\eta_1)\,VU_2$. Applying Π_1 to both sides of the last equation yields $\Pi_0(V)[\Pi_0(T(f))\circ\eta_2]=\Pi_0(f\circ\eta_1)\,\Pi_0(V)$. Since $\Pi_0(V)$ is invertible in the commutative algebra $C(X_2)$, we conclude that $f\circ\tau\circ\eta_2=f\circ\eta_1\circ\tau$ for every f in $C(X_1)$. Thus $\tau\circ\eta_2=\eta_1\circ\tau$. Therefore (X_1,η_1) and (X_2,η_2) are conjugate.

It is clear that if two pairs (X_i,η_i) are conjugate, then every conjugacy algebra of (X_1,η_1) is isomorphic to some conjugacy algebra of (X_2,η_2). Therefore if A is a functor that assigns a conjugacy algebra $A(X,\eta)$ to each pair (X,η) such that $A(X_1,\eta_1)$ is isomorphic to $A(X_2,\eta_2)$ whenever (X_1,η_2) and (X_2,η_2) are conjugate, then there is an Arveson–Josephson type of theorem for A under the hypothesis of Theorem 3.1: $A(X_1,\eta_2)$ is isomorphic to $A(X_2,\eta_2)$ if and only if (X_1,η_1) and (X_2,η_2) are conjugate. Examples of such functors are $P(X,\eta)$, $P^\infty(X,\eta)$, and the Banach algebra $R(X,\eta)$ of operators on $C(X)$ with elements of $C(X)$ acting on $C(X)$ as multiplications and U defined by $Uf=f\circ\eta$, and with $R(X,\eta)$ generated by $C(X)$ and U. Another important example of such a functor is the semi-crossed product algebra in [5]; therefore we obtain J. Peter's extension of the Arveson–Josephson theorem with the aperiodicity condition replaced by the (weaker) assumption on fixed points in Theorem 3.1.

COROLLARY 3.7. *Suppose* (X_i,η_i), $i=1,2$, *satisfy the hypothesis of Theorem* 3.1. *Then* (X_1,η_1) *and* (X_2,η_2) *are conjugate if and only if their semi-crossed product algebras are isomorphic.*

The ideas of the simplified proof in the absence of fixed points show that an Arveson–Josephson type theorem holds for the functor $P^\infty(X,\eta)$ without any assumptions on fixed points. The fact that $P^\infty(X,\eta)\,U$ is the intersection of the kernels of all the complex homorphisms on $P^\infty(X,\eta)$ was shown in the proof of Lemma 2.6.

COROLLARY 3.8. *Two pairs* (X_i,η_i) *are conjugate if and only if* $P^\infty(X_1,\eta_1)$ *and* $P^\infty(X_2,\eta_2)$ *are isomorphic.*

Remark. The ideas of the simplified proof show that if η has no fixed points and A is a conjugacy algebra for (X,η), then AU is the commutator ideal of A, i.e., the ideal generated by $\{ab-ba\colon a,b\in A\}$. The commutator ideal contains $\{(f-f\circ\eta)\,U\colon f\in C(X)\}$ and since the ideal I of $C(X)$ generated by $\{f-f\circ\eta\colon f\in C(X)\}$ does not vanish at some point of X, we have $I = C(X)$. Thus the commutator ideal of A contains U, and therefore AU. The reverse inclusion is obvious since $ab - ba \in AU = \ker \Pi_0$.

In sequels to this paper we develop a theory of conjugacy algebras for pairs (X, η), where η is a continuous map on the locally compact Hausdorff space X, and we prove an analogue of Theorem 3.1. We also prove a version of Theorem 3.1. for certain semigroups of continuous maps, rather than single maps. In addition we prove that the original result of Arveson [1] holds without any ergodicity or aperiodicity assumptions on the $*$-automorphisms of the L^∞-spaces.

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